Integrable systems associated with the Bruhat Poisson structures.

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Abstract

The purpose of this note is to give a simple description of a (complete) family of functions in involution on certain hermitian symmetric spaces. This family obtained via the bi-hamiltonian approach using the Bruhat Poisson structure is especially simple for projective spaces, where the formulas in terms of the momentum map coordinates are presented. We show how these functions are related to the Gelfand-Tsetlin coordinates. We also show how the Lenard scheme can be applied.

1 Introduction.

Let K be a compact real form of a complex semi-simple Lie group G and let $H \subset K$ be a subgroup of K defined by $H = K \cap P$, where P is a parabolic subgroup of G containing a Borel subgroup $B \subset G$. The Bruhat Poisson structure π_∞ on $X = K/H$, first introduced by Soibelman [14] and Lu-Weinstein [8], has the property that its symplectic leaves are precisely the Bruhat cells in X. If $T = K \cap B$ is a maximal torus of K, then π_∞ is T-invariant. Let ω_s (respectively, π_s) stand for a K-invariant symplectic form (respectively, dual bi-vector field) on X, which we assume now to be a compact hermitian symmetric space. It was shown by Khoroshkin-Radul-Rubtsov in [5] that the two Poisson structures, π_∞ and π_s are compatible, meaning that the Schouten bracket of π_∞ and π_s vanishes, $[\pi_\infty, \pi_s] = 0$. In particular, any bi-vector field of the form $\alpha\pi_\infty + \beta\pi_s$, $(\alpha, \beta) \in \mathbb{R}^2$, is Poisson. In this situation, one can introduce the following family $\{f_k\}$ of functions:

$$f_k := ({\pi_\infty}^{\wedge k}, \omega_s^{\wedge k}),$$

obtained by the duality pairing of exterior powers of ω and π_∞. If the (real) dimension of X is equal to $2n$, then we have n functions, f_1, ..., f_n which may carry some useful information about X. These functions are in involution with

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respect to either of the two Poisson structures. We make explicit computations for $\mathbb{CP}^n$, since this is the only case, where we can present an explicit coordinate approach. We show how the function that we have obtained are related to the Gelfand-Tsetlin integrable systems studied by Guillemin and Sternberg [4]. Analogous statements for other hermitian symmetric spaces will appear elsewhere [1]. In the last part of the paper we make explicit computations using the Lenard scheme [11].

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2 Families of functions in involution.

Multi-hamiltonian structures are very important in the theory of integrable systems. Starting with the fundamental works of Magri [11], bi- and multi-Hamiltonian structures found many interesting and fundamental applications, as in [7], [2], [13] and references therein.

Let M be a manifold and let π_b and π_s be two Poisson structures on M such that
1. The Poisson structure π_s is non-degenerate (so the subscript s stands for symplectic).
2. The Poisson structures π_s and π_b are compatible, meaning that the Schouten bracket $[\pi_s, \pi_b]$ vanishes. Or, equivalently, for any two real numbers α and β, the bi-vector field $\alpha\pi_s + \beta\pi_b$ defines a Poisson structure on M.

If $\dim(M) = 2n$, then we can define n functions f_1, ..., f_n as follows:

$$f_j = \frac{\pi_b^j \wedge \pi_s^{n-j}}{\pi_s^n}.$$

The operation of division by the top degree bi-vector field makes perfect sense, since π_s is non-degenerate, and thus in any local coordinate system $(x_1, \ldots\ x_{2n})$ the $2n$-vector field π_s^n looks like

$$\pi_s^n = h(x_1, ..., x_{2n})\partial_{x_1} \wedge \cdots \wedge \partial_{x_{2n}},$$

for a non-vanishing function $h(x_1, ..., x_{2n})$. Equivalently, if ω_s is the symplectic form dual to π_s, then one can define

$$f_k := ({\pi_\infty}^{\wedge k}, \omega_s^{\wedge k}),$$

where we use the duality pairing

$$\Gamma(M, \wedge^{2k}TM) \otimes \Gamma(M, \wedge^{2k}T^*M) \to C^\infty(M).$$

It turns out that this family of functions has the following property.

PROPOSITION 2.1 *The family of functions f_i defined above are in involution with respect to either Poisson structure, π_b or π_s.*

Proof. (**J.-H. Lu**) Let $X_i = i_{df_i}\pi_b$ and let $Y_j = i_{df_j}\pi_s$. Consider the equality $f_k\pi_s^n = \pi_b^k \wedge \pi_s^{n-k}$ and compute L_{X_l} of both sides to arrive to the following identity:

$$\frac{n-k}{k+1}\{f_{k+1}, f_l\}_s = -\{f_k, f_l\}_b + nf_k\{f_1, f_l\}_s,$$

where the subscripts s or b indicate with respect to which Poisson structure the Poisson bracket is taken. Finally, use the induction on l. ◯

Remark. The approach that we have followed here is intimately related to the Poisson-Nijenhuis structures, that were studied by Magri and Morosi [12], Kosmann-Schwarzbach and Magri [7], Vaisman [15] and others. The set of our functions $\{f_j\}$ can be expressed, polynomially, through the traces of powers of the intertwining operator corresponding to the Nijenhuis tensor.

Now let us take $M = X$ to be a coadjoint orbit in $\mathfrak{k}^*$, which we assume to be a compact hermitian symmetric space. We take $\pi_s = \pi$ - the Kirillov-Kostant-Souriau symplectic structure and $\pi_b = \pi_\infty$ - the Bruhat-Poisson structure, which is obtained via an identification of X with $K/(P \cap K)$ as in Introduction. Under this identification, π is K-invariant. The following was first proved in [5]:

PROPOSITION 2.2 *If X is a hermitian symmetric space as above, then the Poisson structures π and π_∞ are compatible.*

Proof. (**J.-H. Lu**) Let X be a generating vector field for the K-action. Clearly, the K-invariance of π implies that $L_X\pi = 0$. Since π_∞ came from $\mathfrak{k} \wedge \mathfrak{k}$ by applying left and right actions of K, $L_X\pi_\infty$ is obtained from $\delta(X)$ by applying the K-action. Here, $\delta(X)$ is the co-bracket of X, which is an element of $\mathfrak{k} \otimes \mathfrak{k}$, since we can view X as an element of $\mathfrak{k}$. Therefore, $L_X\pi_\infty$ is a sum of wedges of generating vector fields for the action of K. Accordingly,

$$[L_X\pi_\infty, \pi] = 0,$$

which in turn implies that

$$L_X[\pi_\infty, \pi] = 0,$$

and thus $[\pi_\infty, \pi]$ is a K-invariant 3-vector field on X. When X is a hermitian symmetric space, there are none such (since the nil-radical of the corresponding parabolic group is abelian), so it must be zero. ◯

Therefore, we have the following

PROPOSITION 2.3 *Let X be a coadjoint orbit in K. Assume that X is a hermitian symmetric space of complex dimension n. The above recipe yields n functions $(f_1, ..., f_n)$ on X, which are in involution with respect to either π_∞ or π.*

The functions $(f_1, ..., f_n)$ that we have constructed turn out to be related to the Gelfand-Tsetlin coordinates in the case when $K = SU(n)$, as we will see later on. In the next section we will carry explicit computations of these functions on the projective spaces.

3 Computations for the projective spaces.

Let $\mathbb{CP}^n$ be a complex projective space of (complex) dimension n, and let $[Z_0 : Z_1 : ... : Z_n]$ be a homogeneous coordinate system on it. We use the standard Fubini-Study form ω for ω_s and the following description of π_∞ obtained by Lu Jiang-Hua in [9] and [10]. First, we need Lu's coordinates on the largest Bruhat cell, where $Z_0 \neq 0$ and we let $z_i = Z_i/Z_0$:

$$y_i := \frac{z_i}{\sqrt{1 + |z_{i+1}|^2 + \cdots + |z_n|^2}}, \quad 1 \leq i \leq n.$$

Lu's coordinates are not holomorphic, but convenient for the Bruhat Poisson structure, which now assumes the following form

$$\pi_\infty = \sqrt{-1} \sum_{i=1}^{n} (1 + |y_i|^2) \partial_{y_i} \wedge \partial_{\bar{y}_i}.$$

In order to be able to compute with ω and π_∞, we need to move to the polar variables r_i, ϕ_j defined by $z_i = r_i e^{\sqrt{-1}\phi_i}$ and eventually to the momentum map variables x_i, ϕ_j defined by

$$x_i = \delta_{1,i} - \frac{r_i^2}{1 + r_1^2 + \cdots r_n^2}.$$

These variable are just a slight distortion (for later convenience) of the standard coordinates on $\mathbb{R}^n$ for the momentum map associated with the maximal compact torus action on $\mathbb{CP}^n$. One of the advantages of using this coordinate system is that the Fubini-Study symplectic structure has the following simple form:

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\phi_i.$$

In fact, the simplest form for the Bruhat Poisson structure is also achieved in this coordinate system.

PROPOSITION 3.1 *The Bruhat Poisson structure π_∞ on $\mathbb{CP}^n$ can be written in the coordinate system (x_j, ϕ_i) as*

$$\pi_\infty = \sum_{i=1}^{n} \Theta_i \wedge \partial_{\phi_i},$$

where

$$\Theta_i = (x_1 + \cdots + x_i)\partial_{x_i} + \sum_{j=i+1}^{n} x_j \partial_{x_j}.$$

Proof. The proof of this statement is purely computational. One can introduce auxiliary variables $q_i = \log(1 + |y_i|^2)$, and use those to write

$$\pi_\infty = \sum_{i=1}^{n} \partial_{q_i} \wedge \partial_{\phi_i}$$

- the action-angle form for π_∞. Eventually, one can establish the following relations: $x_1 = e^{-q_1}$, and for $j > 1$,

$$x_j = e^{-(q_1+\cdots+q_j)} - e^{-(q_1+\cdots+q_{j-1})}.$$

The rest is straightforward. ◯

Now, one can see that the simple linear and triagonal form of π_∞ makes the computation of the functions $\{f_i\}$ extremely simple. We will introduce the following linear change of variables on $\mathbb{R}^n$:

$$c_k = \sum_{i=1}^{k} x_i.$$

In these variables, the set of functions $\{f_i\}$ looks as follows.

THEOREM 3.2 *The integrals f_i (up to constant multiples) arising from the bihamiltonian structure (π_s, π_∞) on $\mathbb{CP}^n$ are given by the elementary polynomials in $(c_1, ..., c_n)$:*

$$f_1 = c_1 + \cdots + c_n,$$

$$f_j = \sum_{i_1 < \ldots < i_j} c_{i_1} \cdots c_{i_j},$$

$$f_n = c_1 \cdots c_n.$$

The explicit nature of these integrals is essential in looking at the relation with the certain natural flows [6]. The hamiltonian f_1 in terms of the momentum map variables is given by

$$f_1 = nx_1 + (n-1)x_2 + \cdots + 2x_{n-1} + x_n.$$

Then the gradient in the momentum simplex has coordinates $\lambda_i = n+1-i$. Those numbers also are the weights assigned to the vertices (which correspond to the centers of the Bruhat cells). Thus we arrive to

THEOREM 3.3 *The above flow on $\mathbb{CP}^n$ with eigenvalues consecutive integers from 1 to n determines the standard Bruhat cell decomposition.*

4 Relation with Gelfand-Tsetlin coordinates.

When $X = Gr(k)$ - the grassmannian of k-planes in $\mathbb{C}^{n+1}$, we have obtained $k(n-k+1)$ functions in involution on X. Let us recall the standard embedding

$$\Psi : \quad F_n \hookrightarrow Gr(1) \times \cdots \times Gr(n),$$

where F_n is the manifold of full flags in $\mathbb{C}^{n+1}$, and the locus of the embedding is given by the incidence relations. This embedding respects the KKS Kaehler structures on the manifolds involved, if we would like to view them as coadjoint orbits in $\mathfrak{k}^*$. Moreover, this embedding is equivariant with respect to the $K = SU(n+1)$-action.

Recall the Gelfand-Tsetlin system on F_n. We fix the orbit type of F_n, i.e. we fix the eigenvalues $\sqrt{-1}\lambda_i$ and order them, so $\lambda_1 > \lambda_2 > \cdots > \lambda_{n+1}$. For convinience and easier visualization, we will assume that $\lambda_{n+1} = 0$ (so all the eigenvalues are non-negative), which will correspond to working with $\mathfrak{u}^*_{n+1}$ rather than with $\mathfrak{su}^*_{n+1}$. For convenience, we also identify the Lie algebra $\mathfrak{u}_k$ with its dual via $-\mathrm{Tr}(AB)$. The Gelfand-Tsetlin system looks like [3], [4]:

$$\begin{array}{ccccccccccc}
\lambda_1 & > & \lambda_2 & > & \cdots & > & \lambda_n & > & 0 \\
 & \mu_1^1 & \geq & \mu_2^1 & \geq & \cdots & \geq & \mu_n^1 & \\
 & & & \cdots & \cdots & \cdots & & & \\
 & & & \mu_1^{n-1} & \geq & \mu_2^{n-1} & & & \\
 & & & & \mu_1^n & & & &
\end{array},$$

where the i-th row corresponds to the projection $\mathfrak{u}^*_{n+1} \to \mathfrak{u}^*_{n+2-i}$, which is dual to the embedding $U(n+2-i) \hookrightarrow U(n+1)$ in the left upper corner. The eigenvalues μ_i^j together with λ_i's satisfy the interlacing property.

The picture above can be adapted to any orbit, in particular to $Gr(k)$, where we would take $\lambda_1 = \cdots = \lambda_k > 0$, and other λ's equal to zero. When k varies from 1 to n, the picture above acquires more and more non-zero elements. At

each step, while going from level k to $k+1$, we will get new integrals on $Gr(k+1)$, which we can pull back to F_n using Ψ.

Our goal is to relate the integrals f_j, that we obtained in Section 3 using the bi-hamiltonian approach on hermitian symmetric spaces, and the Gelfand-Tsetlin coordinates. We will start working with $M = \mathbb{CP}^n$, the complex projective space.

Let B be the $(n+1) \times (n+1)$ matrix, representing an element of $\mathfrak{u}(n+1)^*$ such that the only non-zero element of B is $\sqrt{-1}\lambda$, located in the very left upper place. The coadjoint orbit $\mathcal{O}_B$ of B is isomorphic to $\mathbb{CP}^n$, where the identification goes as follows. Any element in the coadjoint orbit of B can be viewed as ABA^{-1}, where $A \in U(n+1)$. Let (a_{ij}) be the entries of A. Then the identification

$$w: \ \mathcal{O}_B \to \mathbb{CP}^n$$

is given by

$$w(ABA^{-1}) = [a_{11} : a_{21} : \cdots : a_{n+1,1}],$$

in terms of a homogeneous coordinate system $[Z_0 : \cdots : Z_n]$ on $\mathbb{CP}^n$. We suspect that the following is well-known, and in any case, is not hard to compute, that the Gelfand-Tsetlin coordinates are:

$$\mu_r^k = 0 \ \text{ for } \ r \neq 1,$$

$$\mu_1^k = \lambda(x_1 + \cdots + x_{n-k+1}),$$

where $(x_1, ..., x_n)$ are the momentum map coordinates that we used in the previous section. We arrive to the conclusion that the Gelfand-Tsetlin coordinates $\{\mu_1^k\}$ coincide (up to the multiple of λ, which we can assume equal to one) with the coordinates $\{c_k\}$ introduced in the previous section. Now, it remains to notice that the Theorem 3.2 from the previous Section immediately yields

THEOREM 4.1 *The complete family of integrals in involution $\{f_i\}$ on $\mathbb{CP}^n$ obtained using the bi-hamiltonian approach with respect to the Bruhat Poisson structure and an invariant symplectic structure are expressed by the elementary polynomials in the Gelfand-Tsetlin coordinates.*

We prove a similar result for other hermitian symmetric spaces in a forthcoming paper [1].

5 Comparison to the Lenard scheme.

Recall the following result [11]. If $\alpha\pi_0 + \beta\pi_1$ is a Poisson pencil on a manifold M, and V a vector field, preserving this pencil, then there exists a sequence of smooth functions $\{g_i\}$ on M, such that g_1 is the Hamiltonian of V with respect

to π_0 and the vector field of the π_0-hamiltonian f_j is the same as the vector field of the π_1-hamiltonian f_{j+1}:

$$i_{df_j}\pi_0 = i_{df_{j+1}}\pi_1.$$

Moreover, the functions in the family $\{f_j\}$ are in involution with respect to both π_0 and π_1.

Our goal in this section is to show that if we start with $M = \mathbb{CP}^n$, and take the pencil (π_s, π_∞) as before, then there is a natural choice of V on $\mathbb{CP}^n$ leading to a completely integrable systems, and the integrals $\{g_j\}$ in question can be easily expressed in terms of the coordinates $(c_1, ..., c_n)$ that we introduced in Section 3.

It is a matter of a simple computation that if we start with a hamiltonian $g_1 = a_1x_1 + \cdots + a_nx_n$, where $(x_1, ..., x_n)$ are the momentum map coordinates as before, then the corresponding initial vector field V is given by

$$V = i_{dg_1}\pi_\infty = \sum_j [a_j(x_1 + \cdots x_j) + a_{j+1}x_{j+1} + \cdots + a_nx_n]\partial_{\phi_j}.$$

From this, one can compute $g_2 = \sum_j \frac{a_j}{2}x_j^2 + \sum_{l<k} a_kx_lx_k$, etc. An interesting choice for g_1 turns out to be

$$g_1 = c_1 + \cdots + c_n = nx_1 + (n-1)x_2 + \cdots + x_n,$$

which coincides with f_1 from Section 3. The reason for this choice is

PROPOSITION 5.1 *The Lenard scheme associated to the Poisson pencil* (π_∞, π_s) *on* $\mathbb{CP}^n$ *which starts with* $g_1 = c_1 + \cdots + c_n$ *and*

$$V = \sum_j [(n-j+1)(x_1 + \cdots + x_j) + (n-j)x_{j+1} + \cdots + 2x_{n-1} + x_n]\partial_{\phi_j},$$

yields

$$g_k = c_1^k + c_2^k + \cdots + c_n^k,$$

which determines a completely integrable bi-hamiltonian system on $\mathbb{CP}^n$.

Proof. With all the explicit formulas that we have presented in this paper, the proof is a simple computation. ◯

We should remark, that the constants $(a_1, ..., a_n)$ for the first hamiltonian in the Lenard scheme have to be chosen with care for two reasons. First, the computations are not simple for an arbitrary choice. Second, as the next example shows, we do not always arrive to a completely integrable system.

Example. If one takes $g_1 = x_1 + \cdots + x_n$, and $V = \sum_j (x_1 + \cdots + x_n)\partial_{\phi_j}$, then applying the above scheme, one would obtain

$$g_k = (x_1 + \cdots + x_n)^k = (g_1)^k.$$

The differentials of all functions in this family are clearly linearly dependent.

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